High-dimensional Expanders from Kac–Moody–Steinberg Groups

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Motivation

- Expander graphs: sparse yet highly connected
- High-dimensional expanders: generalization to simplicial complexes
- Several non-equivalent ideas: (local) spectral, cosystolic, coboundary, geometric, topological high-dimensional expansion
- Applications for LDPC error-correcting codes (spectral), property testing (cosystolic), non-sofic groups (cosystolic with coefficients in symmetric groups).

Previous constructions: Lubotzky-Samuels-Vishne (2005): Ramanujan complexes; Kaufman-Oppenheim (2018), O'Donnel-Pratt (2022): Coset complexes over (certain) Chevalley groups giving rise to local spectral expanders.

High-dimensional expanders

Let $G = (V, E)$ be a finite, simple, d-regular graph.
Random walk matrix given by $M \in \mathrm{Mat}_{|V|}(\mathbb{R})$:

$$M_{v,w} = \begin{cases} \frac{1}{d} & \text{if } (v,w) \in E \\ 0 & \text{else.} \end{cases}$$

Eigenvalues of M: $\lambda_1 = 1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|}$.

Definition 1 *G is called λ-expander if*

$$\lambda_2(M) \geq \lambda.$$

Let X be a finite *simplicial complex* which is

- d-dimensional, i.e. $\max_{\sigma \in X}(|\sigma| - 1) = \max_{\sigma \in X} \dim(\sigma) = d$,
- pure, i.e. $\forall \tau \in X \exists \sigma \in X(d) : \tau \subseteq \sigma$.

We use the following notation

- $X(i) = \{\sigma \in X \mid \dim(\sigma) = i\}$ for $-1 \leq i \leq d$, note that $X(-1) = \{\emptyset\}$.
- The **link** of $\tau \in X$: $\mathrm{lk}_X(\tau) = \{\sigma \in X \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in X\}$. Note $\dim(\mathrm{lk}_X(\tau)) = d - \dim(\tau) - 1$.

Definition 2 (Oppenheim 2018) *Given a pure, finite, d-dimensional complex X such that, for some $\lambda < \frac{1}{d}$,*

- *the 1-skeleton of X is connected,*
- *for every $\tau \in X$ with $\dim(\tau) \leq d - 2$ we have that the 1-skeleton of $\mathrm{lk}_X(\tau)$ is connected,*
- $\forall \tau \in X(d-2) : \lambda_2(\mathrm{lk}_X(\tau)) \geq \varepsilon$,

then X is a λ-local spectral expander.

The complete simplex is a local spectral expander, but what we want:

Definition 3 *Let $\varepsilon > 0, c, d \in \mathbb{N}$ be fixed. A family $(X_s)_{s \in \mathbb{N}}$ of finite, pure, d-dimensional complexes is a family of high-dimensional expanders if*

- *for all $s \in \mathbb{N}$: X_s is an ε-spectral/coboundary/cosystolic expander,*
- *for all $s \in \mathbb{N}$, $v \in X_s(0)$: $\deg(v) \leq c$,*
- $|X_s(0)| \to \infty$ *for* $s \to \infty$.

Coset complexes

Definition 4 *Let G be a finite group and $H_0, \dots, H_d$ subgroups of G. Then the coset complex $\mathcal{CC}(G, (H_i)_{i=0}^d)$ is a d-dimensional simplicial complex with*

- *vertices $\bigsqcup_{i=0}^d G/H_i$,*
- *maximal faces $\{gH_0, \dots, gH_d\}$ for $g \in G$.*

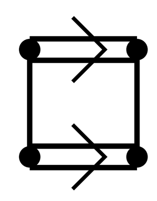
In particular $gH_i \sim hH_j$ if $i \neq j$ and $gH_i \cap hH_j \neq \emptyset$.

Kac–Moody–Steinberg groups

Let k be a (finite) field, A Dynkin diagram with nodes I that is 2-spherical.

Set, for all $i, j \in I, i \neq j$

Example A:



$U_i \cong (k, +)$

if $i \bullet \quad \bullet j$: $U_{ij} = U_i \times U_j$,

if $i \bullet\!\!-\!\!-\!\!\bullet j$: $U_{ij} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \mid a, b, c \in k \right\}$

if $i \bullet\!\Rightarrow\!\bullet j$: $U_{ij} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 \\ ac + d & a & 1 & 0 \\ b & d & -c & 1 \end{pmatrix} \mid x, z, a, c \in k \right\} \leq \mathrm{Sp}_4(k)$

Embedding $f_{i,j} : U_i \to U_{i,j}$ given by $U_i \leftrightarrow a$, $U_j \leftrightarrow c$.

KMS group of type A over k:

$$\mathcal{U}_A(k) = \mathop{\ast}_{i,j \in I} U_{\{i,j\}} \Big/ (U_i \xhookrightarrow{f_{i,j}} U_{ij}; i \neq j \in I)$$
$$= \langle U_i, U_{ij}; i, j \in I \mid \forall i \neq j \in I, \forall a \in U_i : a = f_{i,j}(a) \rangle$$

- A not spherical $\Rightarrow \mathcal{U}_A(k)$ infinite and $U_{ij} \hookrightarrow \mathcal{U}_A(k)$
- For $J \subset I$ let $U_J = \langle U_j \mid j \in J \rangle \leq \mathcal{U}_A(k)$
  If sub-diagram induced by J is spherical, U_J is finite
- in many cases $\mathcal{U}_A(k)$ is residually finite
- if A is affine, $\mathcal{U}_A(k)$ has quotient inside Chevalley groups over $k[t]/(f)$ like in the example.

Main theorem

- $\mathcal{U}_A(k)$ a KMS-group such that
  - k is a finite field, $|k| \geq 4$,
  - A is a Dynkin diagram on nodes I such that any sub-diagram of size $|I| - 1$ is spherical.
- G a finite group, $\phi : \mathcal{U}_A(k) \twoheadrightarrow G$ such that
  - $\phi|_{U_J}$ is injective for all $J \subsetneq I$,
  - $\phi(U_J) \cap \phi(U_K) = \phi(U_J \cap U_K)$ for all $J, K \subsetneq I$.

Then

$$\mathcal{CC}\left(G, (\phi(U_{I \setminus \{i\}}))_{i \in I}\right)$$

is a γ-local spectral expander, where γ is independent of ϕ, G.

Example construction

- Fix a finite field k, $|k| \geq 4$, $\mathrm{char}(k) > 2$.
- Consider the following subgroups inside $\mathrm{SL}_3(k[t])$:

$$H_0 = \begin{pmatrix} 1 & k & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}, H_1 = \begin{pmatrix} 1 & 0 & 0 \\ kt & 1 & k \\ kt & 0 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ kt & kt & 1 \end{pmatrix}.$$

- Irreducible polynomials $f_m \in k[t]$, $m \in \mathbb{N}$, such that $\deg(f_m) \to \infty$.
  $\pi_m : \mathrm{SL}_3(k[t]) \to \mathrm{SL}_3(k[t]/(f_m))$ the entry-wise projection.
- Set $G_m = \mathrm{SL}_3(k[t]/(f_m))$, $H_i^m = \pi_m(H_i)$, $i = 0, 1, 2$

Then

$$(\mathcal{CC}(G_m, (H_0^m, H_1^m, H_2^m)))_{m \in \mathbb{N}}$$

is a family of bounded degree spectral expanders.